

Second-order Chiral Kinetic Theory

高建华 | 山东大学

S. Z. Yang, S. X. Ma, JHG, Phys.Rev.D 112 (2025) e-Print:
2409.15878

第二届旋转与磁场下的 QCD 物质研讨会 | 2026年9月11日-14日 | 安徽淮南

Outline

- Introduction
- Chiral kinetic theory at zeroth and first order
- Chiral kinetic theory at second order
- Summary and outlook

Introduction

Global polarization: First theoretical prediction



Globally polarized quark-gluon plasma in non-central A+A collisions

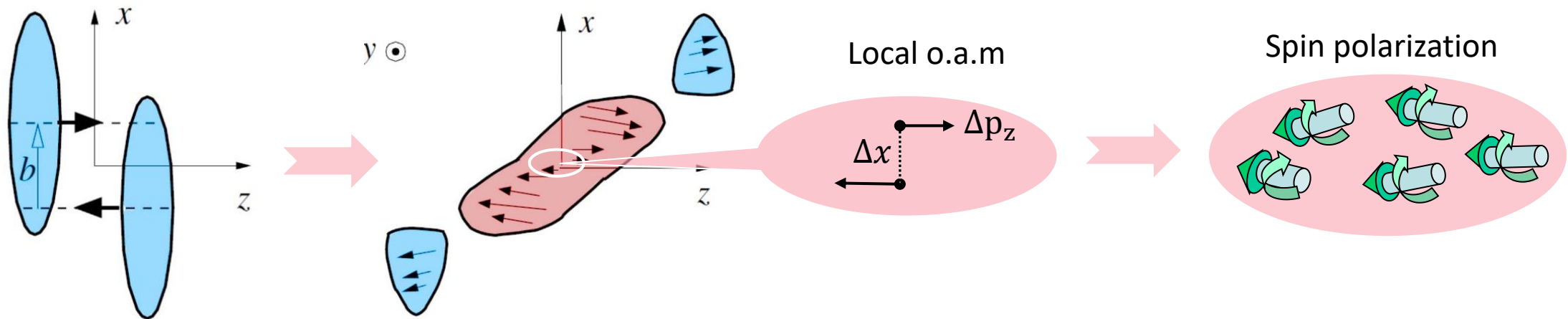
Zuo-Tang Liang (Shandong U.), Xin-Nian Wang (LBL, Berkeley and Shandong U.) (Oct, 2004)

Published in: *Phys.Rev.Lett.* 94 (2005) 102301, *Phys.Rev.Lett.* 96 (2006) 039901 (erratum) • e-Print: [nucl-th/0410079](#) [nucl-th]

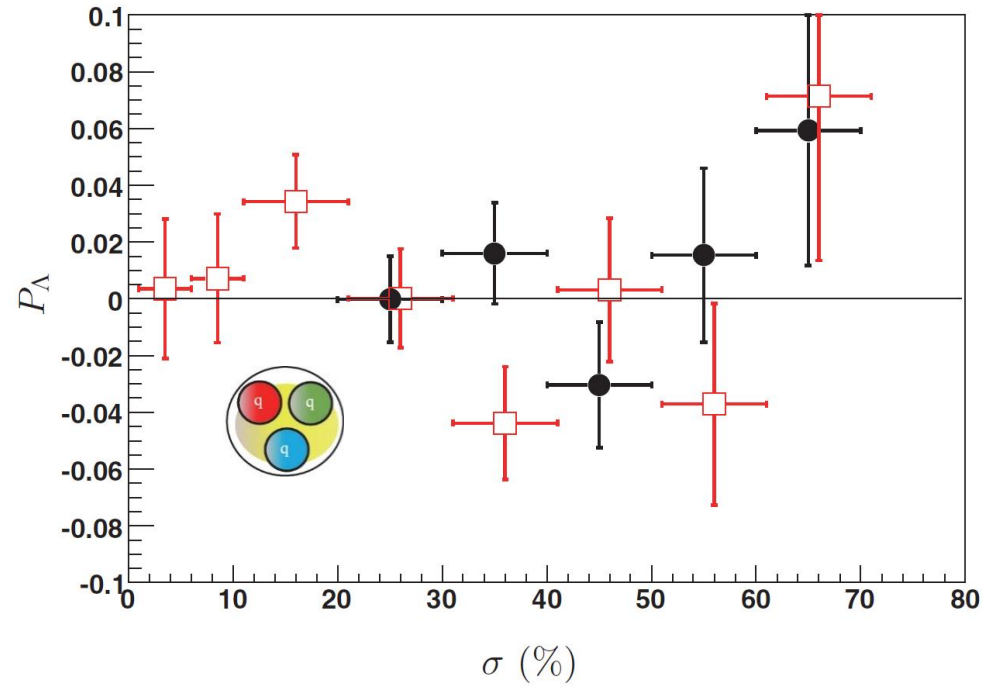
Spin alignment of vector mesons in non-central A+A collisions

Zuo-Tang Liang (Shandong U.), Xin-Nian Wang (LBL, Berkeley and Shandong U.) (Oct, 2004)

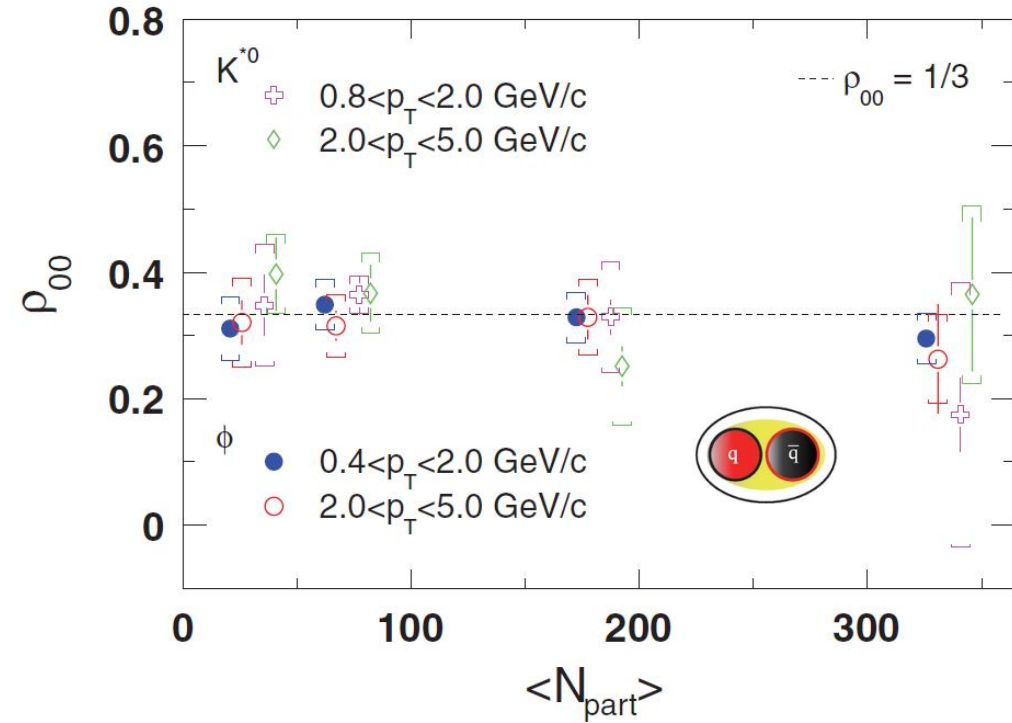
Published in: *Phys.Lett.B* 629 (2005) 20-26 • e-Print: [nucl-th/0411101](#) [nucl-th]



Global polarization: Early experimental results



STAR PRD 76, 024915, 2007



STAR PRC 77, 061902, 2008

Focusing on global polarization: my doctoral research

PHYSICAL REVIEW C 77, 044902 (2008)

Global quark polarization in noncentral $A+A$ collisions

Jian-Hua Gao,¹ Shou-Wan Chen,² Wei-tian Deng,^{1,3} Zuo-Tang Liang,¹ Qun Wang,² and Xin-Nian Wang^{1,3}

¹*Department of Physics, Shandong University, Jinan, Shandong 250100, People's Republic of China*

²*Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, People's Republic of China*

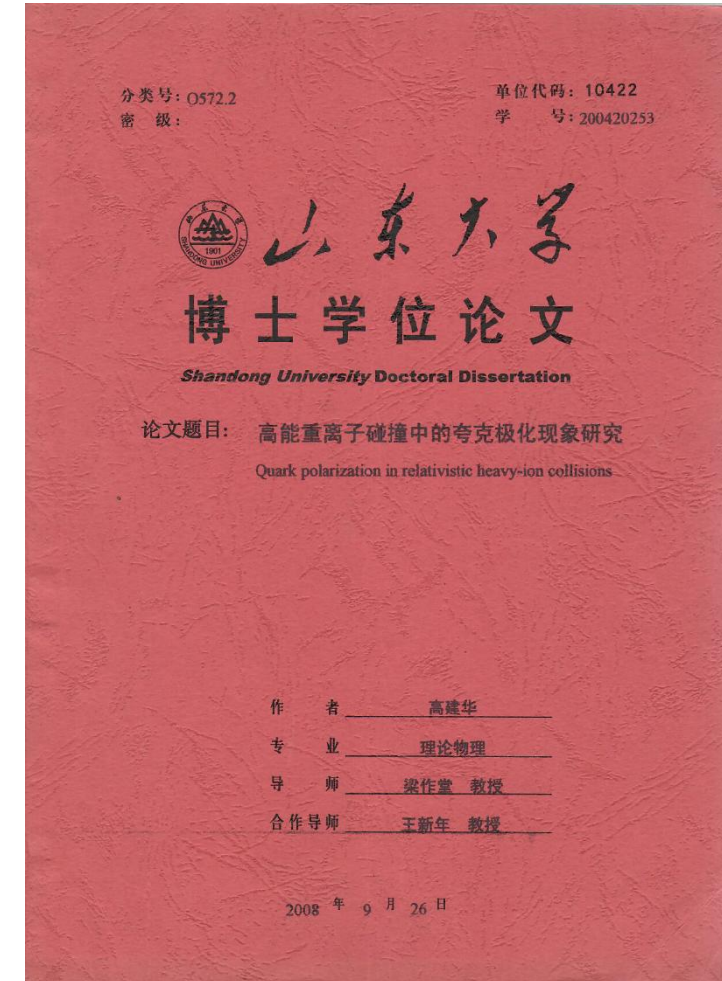
³*Nuclear Science Division, MS 70R0319, Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA*

(Received 12 October 2007; revised manuscript received 28 December 2007; published 9 April 2008)

Partons produced in the early stage of noncentral heavy-ion collisions can develop a longitudinal fluid shear because of the unequal local number densities of participant target and projectile nucleons. Under such fluid shear, local parton pairs with nonvanishing impact parameters have finite local relative orbital angular momentum along the direction opposite to the reaction plane. Such a finite relative orbital angular momentum among locally interacting quark pairs can lead to global quark polarization along the same direction because of spin-orbital coupling. Local longitudinal fluid shear is estimated within both the Landau fireball and the Bjorken scaling model of initial parton production. Quark polarization through quark-quark scatterings with the exchange of a thermal gluon is calculated beyond the small-angle scattering approximation in a quark-gluon plasma. The polarization is shown to have a nonmonotonic dependence on the local relative orbital angular momentum dictated by the interplay between electric and magnetic interactions. It peaks at a value of relative orbital angular momentum which scales with the magnetic mass of the exchanged gluons. With the estimated small longitudinal fluid shear in semiperipheral Au+Au collisions at energies currently available at the BNL Relativistic Heavy Ion Collider (RHIC), the final quark polarization is found to be small $|P_q| < 0.04$ in the weak coupling limit. Possible behavior of the quark polarization in the strong coupling limit and implications on the experimental detection of such global quark polarization at RHIC and CERN Large Hadron Collider (LHC) are also discussed.

DOI: [10.1103/PhysRevC.77.044902](https://doi.org/10.1103/PhysRevC.77.044902)

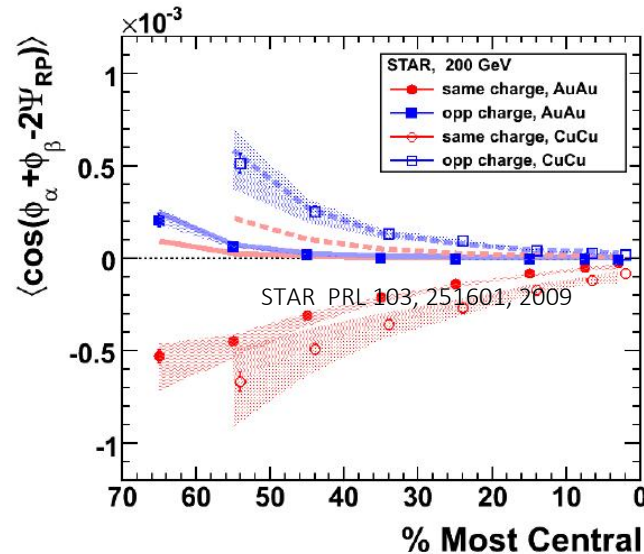
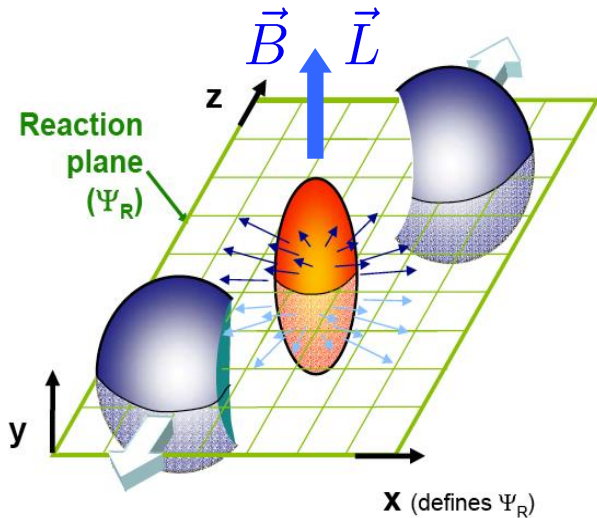
PACS number(s): 25.75.-q, 13.88.+e, 12.38.Mh



Turning to chiral effects: my postdoctoral work

Chiral anomaly: $\partial_\mu J_5^\mu = CE \cdot B$

Chiral magnetic effect: $J^\mu = \frac{\mu_5}{2\pi^2} B^\mu$



Hydrodynamics with Triangle Anomalies

Dam T. Son¹ and Piotr Surówka^{2,3}

¹*Institute for Nuclear Theory, University of Washington, Seattle, Washington 98195-1550, USA*

²*Department of Physics, University of Washington, Seattle, Washington 98195-1560, USA*

³*Institute of Physics, Jagiellonian University, Reymonta 4, 30-059 Kraków, Poland*

(Dated: June 2009)

We consider the hydrodynamic regime of theories with quantum anomalies for global currents. We show that a hitherto discarded term in the conserve current is not only allowed by symmetries, but is in fact required by triangle anomalies and the second law of thermodynamics. This term leads to a number of new effects, one of which is chiral separation in a rotating fluid at nonzero chemical potential. The new kinetic coefficients can be expressed, in a unique fashion, through the anomalies coefficients and the equation of state. We briefly discuss the relevance of this new hydrodynamic term for physical situations, including heavy ion collisions.

a proof similar to the one of the constancy of the entropy/viscosity density ratio [15, 16] is possible. It is also interesting to understand how the new term emerges, for weakly coupled theories, within the kinetic framework. Finally, it is tempting to speculate that Eq. (3) can be derived by direct anomalies matching. At this moment we do not have a complete understanding of the microscopic origin of the vorticity-induced current.

Chiral effects from kinetic theory

PHYSICAL REVIEW D **83**, 094017 (2011)

Consistent description of kinetic equation with triangle anomaly

Shi Pu, Jian-hua Gao, and Qun Wang

*Interdisciplinary Center for Theoretical Study and Department of Modern Physics,
University of Science and Technology of China, Hefei 230026, China*

(Received 24 October 2010; revised manuscript received 4 March 2011; published 13 May 2011)

We provide a consistent description of the kinetic equation with a triangle anomaly which is compatible with the entropy principle of the second law of thermodynamics and the charge/energy-momentum conservation equations. In general an anomalous source term is necessary to ensure that the equations for the charge and energy-momentum conservation are satisfied and that the correction terms of distribution functions are compatible to these equations. The constraining equations from the entropy principle are derived for the anomaly-induced leading order corrections to the particle distribution functions. The correction terms can be determined for the minimum number of unknown coefficients in one charge and two charge cases by solving the constraining equations.

An anomalous source term must be introduced in classical Boltzmann equation:

$$p^\mu \left(\partial_\mu - F_{\mu\nu} \partial_p^\nu \right) f(x, p) = \mathcal{C}[f] + \mathcal{S}_A(x, p)$$

$$\int \frac{d^3p}{(2\pi)^3 E_p} \mathcal{S}_A(x, p) = -CE \cdot B \quad \int \frac{d^3p}{(2\pi)^3 E_p} p^\mu \mathcal{S}_A(x, p) = 0$$

PRL **109**, 232301 (2012)

PHYSICAL REVIEW LETTERS

week ending
7 DECEMBER 2012

Chiral Anomaly and Local Polarization Effect from the Quantum Kinetic Approach

Jian-Hua Gao,^{1,2} Zuo-Tang Liang,³ Shi Pu,² Qun Wang,² and Xin-Nian Wang^{4,5}

¹*School of Space Science and Physics, Shandong University at Weihai, Weihai 264209, China*

²*Interdisciplinary Center for Theoretical Study and Department of Modern Physics,
University of Science and Technology of China, Hefei 230026, China*

³*School of Physics, Shandong University, Jinan, Shandong 250100, China*

⁴*Key Laboratory of Quark and Lepton Physics (MOE) and Institute of Particle Physics,
Central China Normal University, Wuhan 430079, China*

⁵*Nuclear Science Division, MS 70R0319, Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA*
(Received 4 March 2012; published 4 December 2012)

A power expansion scheme is set up to determine the Wigner function that satisfies the quantum kinetic equation for spin-1/2 charged fermions in a background electromagnetic field. Vector and axial-vector current induced by magnetic field and vorticity are obtained simultaneously from the Wigner function. The chiral magnetic and vortical effect and chiral anomaly are shown as natural consequences of the quantum kinetic equation. The axial-vector current induced by vorticity is argued to lead to a local polarization effect along the vorticity direction in heavy-ion collisions.

Chiral effects emerge naturally in quantum kinetic theory by Wigner function:

$$\begin{aligned} p^\mu \mathcal{V}_\mu &= 0, & p^\mu \mathcal{A}_\mu &= 0, \\ \nabla^\mu \mathcal{V}_\mu &= 0, & \nabla^\mu \mathcal{A}_\mu &= 0, \\ \epsilon_{\mu\nu\rho\sigma} \nabla^\rho \mathcal{A}^\sigma &= 2(p_\mu \mathcal{V}_\nu - p_\nu \mathcal{V}_\mu), \\ \epsilon_{\mu\nu\rho\sigma} \nabla^\rho \mathcal{V}^\sigma &= 2(p_\mu \mathcal{A}_\nu - p_\nu \mathcal{A}_\mu). \end{aligned} \quad \begin{aligned} \partial_\mu J_5^\mu &= -\frac{1}{4\pi^2} E \cdot B \\ J^\mu &= \frac{\mu_5}{2\pi^2} B^\mu \end{aligned}$$

Berry monopole as source term for chiral effects

PRL **109**, 162001 (2012) PHYSICAL REVIEW LETTERS week ending 19 OCTOBER 2012

Chiral Kinetic Theory

M.A. Stephanov and Y. Yin

Department of Physics, University of Illinois, Chicago, Illinois 60607, USA
(Received 11 July 2012; published 19 October 2012)

We derive the nonequilibrium kinetic equation describing the motion of chiral massless particles in the regime where it can be considered classically. We show that the Berry monopole which appears at the origin of the momentum space due to level crossing is responsible for the chiral magnetic and vortical effects.

PHYSICAL REVIEW D **87**, 085016 (2013)

Kinetic theory with Berry curvature from quantum field theories

Dam Thanh Son¹ and Naoki Yamamoto^{2,3}

¹Enrico Fermi Institute, University of Chicago, 5640 South Ellis Avenue, Chicago, Illinois 60637, USA
²Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan
³Maryland Center for Fundamental Physics, Department of Physics, University of Maryland, College Park, Maryland 20742-4111, USA
(Received 31 January 2013; published 8 April 2013)

A kinetic theory can be modified to incorporate triangle anomalies and the chiral magnetic effect by taking into account the Berry curvature flux through the Fermi surface. We show how such a kinetic theory can be derived from underlying quantum field theories. Using the new kinetic theory, we also compute the parity-odd correlation function that is found to be identical to the result in the perturbation theory in the next-to-leading order hard dense loop approximation.

PRL **110**, 262301 (2013) PHYSICAL REVIEW LETTERS week ending 28 JUNE 2013

Berry Curvature and Four-Dimensional Monopoles in the Relativistic Chiral Kinetic Equation

Jiunn-Wei Chen,¹ Shi Pu,^{1,2} Qun Wang,² and Xin-Nian Wang^{3,4}

¹Department of Physics, National Center for Theoretical Sciences, and Leung Center for Cosmology and Particle Astrophysics, National Taiwan University, Taipei 10617, Taiwan
²Interdisciplinary Center for Theoretical Study and Department of Modern Physics, University of Science and Technology of China, Hefei 230026, China
³Key Laboratory of Quark and Lepton Physics (MOE) and Institute of Particle Physics, Central China Normal University, Wuhan, 430079, China
⁴Nuclear Science Division, MS 70R0319, Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA
(Received 16 March 2013; published 24 June 2013)

We derive a relativistic chiral kinetic equation with manifest Lorentz covariance from Wigner functions of spin-1/2 massless fermions in a constant background electromagnetic field. It contains vorticity

The action with Berry connection: $S = \int dt \left[\vec{p} \cdot \dot{\vec{r}} + \vec{A} \cdot \dot{\vec{r}} - \vec{A}_p \cdot \dot{\vec{p}} - \epsilon_p - A^0 \right] \quad \vec{A}_p = -iu_p^\dagger \vec{\nabla}_p u_p$

Berry monopole in momentum space: $\vec{\Omega}_p = \vec{\nabla}_p \times \vec{A}_p = \frac{s\vec{p}}{2p^2} \quad \vec{\nabla}_p \cdot \vec{\Omega}_p = 2\pi s \delta^3(\vec{p})$

Chiral kinetic equation: $\partial_t f + \frac{1}{(1 + \vec{B} \cdot \vec{\Omega}_p)} \left\{ \left[\vec{v} + \vec{E} \times \vec{\Omega}_p + (\vec{v} \cdot \vec{\Omega}_p) \vec{B} \right] \cdot \vec{\nabla}_x f + \left[\vec{E} + \vec{v} \times \vec{B} + (\vec{E} \cdot \vec{B}) \vec{\Omega}_p \right] \cdot \vec{\nabla}_p f \right\} = 0$

Effective energy and velocity: $\epsilon_p = p - \frac{\vec{B} \cdot \vec{p}}{2p} \quad \vec{v} = \vec{\nabla}_p \epsilon_p \quad \text{Invariant phase space: } d\Gamma = \sqrt{\omega} \frac{d^3 \vec{p} d^3 \vec{x}}{(2\pi)^3} \quad \sqrt{\omega} = (1 + \vec{B} \cdot \vec{\Omega}_p)$

Chiral magnetic effect: $\vec{j} = \frac{1}{4\pi^2} \mu_5 \vec{B} \cdot f(\vec{p} = 0) \quad \text{Chiral anomaly: } \partial_t n + \vec{\nabla}_x \cdot \vec{j} = \frac{1}{4\pi^2} \vec{E} \cdot \vec{B} \cdot f(\vec{p} = 0)$

Chiral kinetic theory by Wigner function approach

PHYSICAL REVIEW D **95**, 091901(R) (2017)
Relativistic chiral kinetic theory from quantum field theories
 Yoshimasa Hidaka,¹ Shi Pu,² and Di-Lun Yang¹
¹Theoretical Research Division, Nishina Center, RIKEN, Wako, Saitama 351-0198, Japan
²Department of Physics, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
 (Received 1 January 2017; published 2 May 2017)
 The chiral kinetic theory of Weyl fermions with collisions in the presence of weak electric and magnetic fields is derived from quantum field theories. It is found that the side-jump terms in the perturbative solution of Wigner functions play a significant role for the derivation. Moreover, such terms manifest the breaking of Lorentz symmetry for distribution functions. The Lorentz covariance of Wigner functions thus leads to modified Lorentz transformation associated with side-jump phenomena further influenced by background fields and collisions.

PHYSICAL REVIEW D **98**, 036010 (2018)
Complete and consistent chiral transport from Wigner function formalism
 Anping Huang,¹ Shuzhe Shi,² Yin Jiang,³ Jinfeng Liao,^{2,*} and Pengfei Zhuang^{1,†}
¹Physics Department, Tsinghua University, Beijing 100084, China
²Physics Department and Center for Exploration of Energy and Matter, Indiana University, 2401 N Milo B. Sampson Lane, Bloomington, Indiana 47408, USA
³School of Physics and Nuclear Energy Engineering, Beihang University, Beijing 100191, China
 (Received 2 February 2018; published 15 August 2018)

PHYSICAL REVIEW D **99**, 085014 (2019)
Chiral kinetic theory in curved spacetime
 Yu-Chen Liu,¹ Lan-Lan Gao,¹ Kazuya Mameda,^{1,*} and Xu-Guang Huang^{1,2,†}
¹Physics Department and Center for Particle Physics and Field Theory, Fudan University, Shanghai 200433, China
²Key Laboratory of Nuclear Physics and Ion-beam Application (MOE), Fudan University, Shanghai 200433, China
 (Received 3 January 2019; published 30 April 2019)

PHYSICAL REVIEW D **101**, 034006 (2020)
Chiral kinetic theory from Landau level basis
 Shu Lin* and Lixin Yang[✉]
 School of Physics and Astronomy, Sun Yat-Sen University, Zhuhai 519082, China
 (Received 4 October 2019; accepted 23 January 2020; published 10 February 2020)
 We derive a chiral kinetic theory with Landau level basis, which is valid for slow-varying magnetic field with arbitrary magnitude. We apply the new chiral kinetic theory to calculate the electric conductivity transverse to the magnetic field in a magnetized QED and QCD plasma. Under the lowest Landau level approximation and relaxation time approximation, we find the transverse conductivity is inversely

PHYSICAL REVIEW D **98**, 036019 (2018)

Disentangling covariant Wigner functions for chiral fermions

Jian-Hua Gao,¹ Zuo-Tang Liang,² Qun Wang,³ and Xin-Nian Wang^{4,5}

4 functions 8 equations $\xrightarrow{O(\hbar^n)}$ 1 function 1 equation

PHYSICAL REVIEW D **101**, 096015 (2020)

Dirac sea and chiral anomaly in the quantum kinetic theory

Jian-Hua Gao,¹ Zuo-Tang Liang,² and Qun Wang³

$$W(x, p) \sim \sum_s c_s \langle a_s^\dagger a_s \rangle + \sum_s \bar{c}_s \left(\langle b_s^\dagger b_s \rangle + \bar{f}_v \right) \quad \bar{f}_v = -1$$

Chiral kinetic theory by Wigner function approach

PHYSICAL REVIEW D **98**, 036019 (2018)

Disentangling covariant Wigner functions for chiral fermions

Jian-Hua Gao,¹ Zuo-Tang Liang,² Qun Wang,³ and Xin-Nian Wang^{4,5}

7D chiral kinetic theory at $\mathcal{O}(\hbar^1)$

$$\begin{aligned} & (1 + \hbar s Q \mathbf{\Omega}_p \cdot \mathbf{B}) \partial_t f(x, E_p, \mathbf{p}) \\ & + \left[\mathbf{v} + \hbar s Q (\mathbf{E} \times \mathbf{\Omega}_p) + \hbar s Q \frac{1}{2|\mathbf{p}|^2} \mathbf{B} \right] \cdot \nabla_x f(x, E_p, \mathbf{p}) \\ & + [Q \tilde{\mathbf{E}} + Q \mathbf{v} \times \mathbf{B} + \hbar s Q^2 (\mathbf{E} \cdot \mathbf{B}) \mathbf{\Omega}_p] \cdot \nabla_p f(x, E_p, \mathbf{p}) = 0, \end{aligned}$$

8D chiral kinetic theory at $\mathcal{O}(\hbar^2)$

$$\begin{aligned} & p_\mu G_{(0)}^\mu [f \delta(\tilde{p}^2)] + \frac{\hbar s}{2} \mathbf{G}^{(0)} \cdot \left\{ \frac{1}{p_0} \mathbf{G}^{(0)} \times [\mathbf{p} f \delta(\tilde{p}^2)] \right\} + \hbar^2 C(f) = 0, \\ & C(f) = \frac{1}{4} \left(p^\mu G_{(0)}^\mu \frac{\mathbf{p}}{p^2} + \mathbf{G}^{(0)} \right) \cdot \left[\frac{1}{p_0} \mathbf{G}^{(0)} \times \left(\frac{1}{p_0} \mathbf{G}^{(0)} \times \mathbf{p} f \delta(p^2) \right) \right] \\ & + \left(p^\mu G_{(0)}^\mu \frac{\mathbf{p}}{p^2} + \mathbf{G}^{(0)} \right) \cdot \left[\frac{1}{p_0} (\mathbf{\Pi}^{(2)} p_0 - \mathbf{\Pi}_0^{(2)} \mathbf{p}) f \delta(p^2) \right] \\ & - p^\mu G_{(0)}^\mu \left[\frac{1}{p^2} \mathbf{\Pi}_\nu^{(2)} p^\nu f \delta(p^2) \right] + G_{(2)}^\mu p^\mu f \delta(p^2). \\ & G_{(0)}^\mu + G_{(2)}^\mu \equiv \partial_x^\mu - Q F^{\mu\nu} \partial_\nu^p + \frac{Q}{24} \hbar^2 \Delta^2 F^{\mu\nu} \partial_\nu^p. \end{aligned}$$

PHYSICAL REVIEW D **112**, 016001 (2025)

Second-order chiral kinetic theory from the Wigner function approach

Shi-Zheng Yang,^{1,2,*} Shu-Xiang Ma^{3,†} and Jian-Hua Gao^{4,5,‡}

8D chiral kinetic theory $\xRightarrow{\mathcal{O}(\hbar^2)}$ 7D chiral kinetic theory

Chiral Wigner functions and equations order by order

Chiral Wigner function: $\mathcal{J}_s^\mu \equiv \frac{1}{8} \int \frac{d^4 y}{(2\pi)^4} e^{-ip \cdot y} \left\langle \bar{\psi} \left(x + \frac{y}{2} \right) \gamma^\mu (1 + s \gamma^5) U \left(x + \frac{y}{2}, x - \frac{y}{2} \right) \psi \left(x - \frac{y}{2} \right) \right\rangle$ Right/Left ($s = \pm 1$)

Chiral Wigner equation: $G_\mu \mathcal{J}_s^\mu = 0, \quad \Pi_\mu \mathcal{J}_s^\mu = 0, \quad 2(\Pi_\mu \mathcal{J}_{s\nu} - \Pi_\nu \mathcal{J}_{s\mu}) = -s \epsilon_{\mu\nu\rho\sigma} G^\rho \mathcal{J}_s^\sigma$ $G^\mu \equiv \partial_x^\mu - j_0 \left(\frac{\partial^p \cdot \partial_x}{2} \right) F^{\mu\nu} \partial_\nu^p$ $\Pi^\mu \equiv p^\mu - \frac{1}{2} j_1 \left(\frac{\partial^p \cdot \partial_x}{2} \right) F^{\mu\nu} \partial_\nu^p$

Semiclassical $\hbar$ expansion: $\mathcal{J}_s^\mu = \sum_{k=0}^{\infty} \hbar^k \mathcal{J}_s^{\mu(k)}$ $G^\mu = \sum_{k=0}^{\infty} \hbar^k G^{\mu(k)}$ $\Pi^\mu = \sum_{k=0}^{\infty} \hbar^k \Pi^{\mu(k)}$

Chiral Wigner equations order by order: $\sum_{k=0}^n G_\mu^{(k)} \mathcal{J}_s^{(n-k)\mu} = 0, \quad \sum_{k=0}^n \Pi_\mu^{(k)} \mathcal{J}_s^{(n-k)\mu} = 0, \quad 2 \sum_{k=0}^n (\Pi_\mu^{(k)} \mathcal{J}_{s\nu}^{(n-k)} - \Pi_\nu^{(k)} \mathcal{J}_{s\mu}^{(k)}) = -s \epsilon_{\mu\nu\rho\sigma} \sum_{k=0}^{n-1} G^{(k)\rho} \mathcal{J}_s^{(n-1-k)\sigma}$

Chiral kinetic theory at zeroth and first order

Zeroth-order Wigner functions from 8D to 7D

General space-time decomposition by introducing n^μ with normalization $n^2 = 1$:

$$X_\mu = X_n n^\mu + \bar{X}^\mu \quad F^{\mu\nu} = E^\mu n^\nu - E^\nu n^\mu - \bar{\epsilon}^{\mu\nu\sigma} B_\sigma \quad \bar{\epsilon}^{\mu\nu\sigma} \equiv \epsilon^{\mu\rho\nu\sigma} n_\rho$$

Independent Wigner functions and on-shell conditions: $\bar{\mathcal{J}}^{(0)\mu} = \frac{1}{p_n} \bar{p}^\mu \mathcal{J}_n^{(0)} \quad \mathcal{J}_n^{(0)} = p_n \mathcal{J}_n^{(0)} \delta(p^2)$

7D Wigner functions: $\int dp_n \bar{\mathcal{J}}^{(0)\mu} = \frac{1}{2} \sum_\lambda \sqrt{\omega^{(0)}} \dot{x}^{(0)\mu} \mathcal{J}_n^{(0)} \quad \int dp_n p_n \mathcal{J}_n^{(0)} = \frac{1}{2} \sum_\lambda \sqrt{\omega^{(0)}} \varepsilon_n^{(0)} \mathcal{J}_n^{(0)}$

Positive/Negative energy:

$$\lambda = +1 \quad \lambda = -1$$

Effective energy:

$$\varepsilon_n^{(0)} = |\bar{p}|$$

Effective four-velocity:

$$\dot{x}^{(0)\mu} = \lambda n^\mu + \hat{\bar{p}}^\mu$$

Invariant phase space factor:

$$\sqrt{\omega^{(0)}} = 1$$

Zeroth-order chiral kinetic equation from 8D to 7D

Integrate 8D Wigner equation with arbitrary weight function: $\int dp_n g(p_n) \nabla_\mu \mathcal{J}^{(0)\mu} = 0 \quad \mathcal{J}^{(0)\mu} = p^\mu \mathcal{J}_n^{(0)} \delta(p^2)$

Pull ∇_μ out of the integral: $\nabla_\mu \int dp_n g(p_n) \mathcal{J}^{(0)\mu} + E_\mu \int dp_n g'(p_n) \mathcal{J}^{(0)\mu} = 0 \quad \nabla^\mu = \partial_\mu^x + E^\mu \bar{\partial}_\mu^p - \bar{\epsilon}^{\mu\rho\nu} B_\rho \bar{\partial}_\nu^p$

The final result after integration: $\sum_\lambda g(\lambda|\bar{p}|) \nabla_\mu \left[\sqrt{\omega^{(0)}} \dot{x}^{(0)\mu} \mathcal{J}_n^{(0)} \right] = 0$

The arbitrary function $g(\lambda|\bar{p}|)$ require : $\dot{x}^{(0)\mu} \nabla_\mu \mathcal{J}_n^{(0)} = 0$ holds for $\lambda = \pm 1$ separately

First-order Wigner functions from 8D to 7D

Independent Wigner functions:

$$\bar{\mathcal{J}}^{(1)\mu} = \frac{1}{p_n} \bar{p}^\mu \mathcal{J}_n^{(1)} + \frac{s}{2p_n} \bar{\epsilon}^{\mu\rho\sigma} \nabla_\rho \mathcal{J}_\sigma^{(0)}$$

Modified on-shell condition:

$$\mathcal{J}_n^{(1)} = p_n \tilde{\mathcal{J}}_n^{(1)} \delta(p^2) + s(B \cdot p) \mathcal{J}_n^{(0)} \frac{\delta(p^2)}{p^2}$$

7D Wigner functions:

$$\int dp_n \mathcal{J}^{(1)\mu} = \frac{1}{2} \sum_\lambda \left[\sqrt{\omega}^{(0)} \dot{x}^{(0)\mu} \tilde{\mathcal{J}}_n^{(1)} + \sqrt{\omega}^{(1)} \dot{x}^{(0)\mu} \mathcal{J}_n^{(0)} + \sqrt{\omega}^{(0)} \dot{x}^{(1)\mu} \mathcal{J}_n^{(0)} \right]$$

$$\int dp_n p_n \mathcal{J}_n^{(1)} = \frac{1}{2} \sum_\lambda \left(\sqrt{\omega}^{(0)} \epsilon^{(0)} \tilde{\mathcal{J}}_n^{(1)} + \sqrt{\omega}^{(0)} \epsilon^{(1)} \tilde{\mathcal{J}}_n^{(0)} + \sqrt{\omega}^{(1)} \epsilon^{(0)} \mathcal{J}_n^{(0)} \right)$$

Wigner function redefinition

$$\mathcal{J}_n^{(1)} \equiv \tilde{\mathcal{J}}_n^{(1)} + \frac{s(B \cdot \bar{p})}{2|\bar{p}|^2} \mathcal{J}_n^{(0)'} \quad \mathcal{J}_n^{(0)'} \equiv \left. \frac{\partial \mathcal{J}_n^{(0)}}{\partial p_n} \right|_{p_n = \lambda |\bar{p}|},$$

$$\sqrt{\omega}^{(1)} = -\frac{s\lambda(B \cdot \bar{p})}{2|\bar{p}|^3} \quad \epsilon^{(1)} = \frac{s\lambda(B \cdot \bar{p})}{2|\bar{p}|^2} \quad \dot{x}^{(1)\mu} = -\frac{s\lambda(B \cdot \bar{p})}{2|\bar{p}|^3} \bar{p}^\mu - \frac{s}{2|\bar{p}|^3} \bar{\epsilon}^{\mu\rho\sigma} E_\rho \bar{p}_\sigma + \frac{s\lambda}{2} \bar{\epsilon}^{\mu\rho\sigma} \bar{\nabla}_\rho \frac{\bar{p}_\sigma}{|\bar{p}|^2} \quad \text{operator}$$

First-order chiral kinetic equation from 8D to 7D

Integrate 8d chiral kinetic equation with arbitrary weight function:

$$\int dp_n g(p_n) \nabla_\mu \mathcal{J}^{(1)\mu} = 0 \quad \mathcal{J}^{(1)\mu} = p^\mu \left[\tilde{\mathcal{J}}_n^{(1)} \delta(p^2) + \frac{sB \cdot p}{p_n} \mathcal{J}_n^{(0)} \frac{\delta(p^2)}{p^2} \right] + \frac{s}{2p_n} \bar{\epsilon}^{\mu\rho\sigma} \nabla_\rho \left[p_\sigma \mathcal{J}_n^{(0)} \delta(p^2) \right]$$

Pull ∇_μ out of the integral: $\nabla_\mu \int dp_n g(p_n) \mathcal{J}^{(1)\mu} + E_\mu \int dp_n g'(p_n) \mathcal{J}^{(1)\mu} = 0$

The final result after integration: $\sum_\lambda g(\lambda|\bar{p}) \nabla_\mu \left[\sqrt{\omega}^{(0)} \dot{x}^{(0)\mu} J_n^{(1)} + \sqrt{\omega}^{(1)} \dot{x}^{(0)\mu} J_n^{(0)} + \sqrt{\omega}^{(0)} \dot{x}^{(1)\mu} J_n^{(0)} \right] = 0$

The arbitrary function $g(\lambda|\bar{p})$ require : $\dot{x}^{(0)\mu} \nabla_\mu \mathcal{J}_n^{(1)} + \dot{x}^{(1)\mu} \nabla_\mu \mathcal{J}_n^{(0)} + \dot{x}^{(0)\mu} [\nabla_\mu, \sqrt{\omega}^{(1)}] \mathcal{J}_n^{(0)} + [\nabla_\mu, \dot{x}^{(1)\mu}] \mathcal{J}_n^{(0)} = 0$

$$\dot{x}^{(0)\mu} \nabla_\mu \mathcal{J}_n^{(1)} - \frac{sB \cdot \bar{p}}{2|\bar{p}|^3} \bar{p}^\mu \bar{\nabla}_\mu \mathcal{J}_n^{(0)} - \frac{\lambda s}{2|\bar{p}|^3} \bar{\epsilon}^{\mu\rho\sigma} E_\rho \bar{p}_\sigma \bar{\nabla}_\mu \mathcal{J}_n^{(0)} = 0$$

Second-order chiral kinetic theory

Second-order Wigner functions from 8D to 7D

Independent Wigner functions:

$$\bar{\mathcal{J}}^{(2)\mu} = \frac{1}{p_n} \bar{p}^\mu \mathcal{J}_n^{(2)} + \frac{s}{2p_n} \bar{\epsilon}^{\mu\rho\sigma} \nabla_\rho \mathcal{J}_\sigma^{(1)} \quad \mathcal{J}^{(1)\mu} = \frac{1}{p_n} p^\mu \mathcal{J}_n^{(1)} + \frac{s}{2p_n} \bar{\epsilon}^{\mu\rho\sigma} \nabla_\rho \mathcal{J}_\sigma^{(0)}$$

Modified on-shell condition:

$$\mathcal{J}_n^{(2)} = p_n \tilde{\mathcal{J}}_n^{(2)} \delta(p^2) + \frac{sB \cdot p}{p^2} \left[\tilde{\mathcal{J}}_n^{(1)} \delta(p^2) + \frac{sB \cdot p}{p_n} \mathcal{J}_n^{(0)} \frac{\delta(p^2)}{p^2} \right] + \frac{s}{2p^2} (\bar{p}_\mu \bar{\nabla}_\rho - \bar{p}_\rho \bar{\nabla}_\mu) \left\{ \frac{s}{2p_n} \nabla^\mu \left[p^\rho \mathcal{J}_n^{(0)} \delta(p^2) \right] \right\}$$

7D Wigner functions:

$$\int dp_n \mathcal{J}^{(2)\mu} = \frac{1}{2} \left[\sqrt{\omega^{(0)}} \dot{x}^{(0)\mu} \tilde{\mathcal{J}}_n^{(2)} + \sqrt{\omega^{(1)}} \dot{x}^{(0)\mu} \mathcal{J}_n^{(1)} + \sqrt{\omega^{(0)}} \dot{x}^{(1)\mu} \mathcal{J}_n^{(1)} + \sqrt{\omega^{(1)}} \dot{x}^{(1)\mu} \mathcal{J}_n^{(0)} + \sqrt{\omega^{(0)}} \dot{x}^{(2)\mu} \mathcal{J}_n^{(0)} + \sqrt{\omega^{(2)}} \dot{x}^{(0)\mu} \mathcal{J}_n^{(0)} \right]$$

$$\int dp_n p_n \mathcal{J}_n^{(2)} = \frac{1}{2} \left[\sqrt{\omega^{(0)}} \varepsilon^{(0)} \tilde{\mathcal{J}}_n^{(2)} + \sqrt{\omega^{(1)}} \varepsilon^{(0)} \mathcal{J}_n^{(1)} + \sqrt{\omega^{(0)}} \varepsilon^{(1)} \mathcal{J}_n^{(1)} + \sqrt{\omega^{(1)}} \varepsilon^{(1)} \mathcal{J}_n^{(0)} + \sqrt{\omega^{(0)}} \varepsilon^{(2)} \mathcal{J}_n^{(0)} + \sqrt{\omega^{(2)}} \varepsilon^{(0)} \mathcal{J}_n^{(0)} \right]$$

$$\begin{aligned} \mathcal{J}_n^{(2)} = & \tilde{\mathcal{J}}_n^{(2)} + \frac{sB \cdot \bar{p}}{2|\bar{p}|^2} \tilde{\mathcal{J}}_n^{(1)'} + \frac{1}{2} \left(\frac{sB \cdot \bar{p}}{2|\bar{p}|^2} \right)^2 \mathcal{J}_n^{(0)''} + \frac{1}{24} (B \cdot \bar{p})^2 \left(\frac{\lambda}{|\bar{p}|^3} \mathcal{J}_n^{(0)'''} - \frac{9\lambda}{|\bar{p}|^5} \mathcal{J}_n^{(0)'} \right) + \frac{1}{24} B^2 \left(\frac{\lambda}{|\bar{p}|} \mathcal{J}_n^{(0)'''} - \frac{3}{|\bar{p}|^2} \mathcal{J}_n^{(0)''} - \frac{3\lambda}{|\bar{p}|^3} \mathcal{J}_n^{(0)'} \right) + \frac{1}{24} [(E \cdot \bar{p})^2 - E^2 \bar{p}^2] \left(\frac{\lambda}{|\bar{p}|^3} \mathcal{J}_n^{(0)'''} - \frac{3}{|\bar{p}|^4} \mathcal{J}_n^{(0)''} + \frac{3\lambda}{|\bar{p}|^5} \mathcal{J}_n^{(0)'} \right) \\ & + \frac{1}{12} \bar{\epsilon}^{\rho\mu\nu} \bar{p}_\rho E_\mu B_\nu \left(\frac{1}{|\bar{p}|^2} \mathcal{J}_n^{(0)'''} - \frac{3\lambda}{|\bar{p}|^3} \mathcal{J}_n^{(0)''} + \frac{3}{|\bar{p}|^4} \mathcal{J}_n^{(0)'} \right) - \frac{1}{8} [(E \cdot \bar{p}) \bar{p}^\mu - \bar{p}^2 E^\mu + \lambda |\bar{p}| \bar{\epsilon}^{\mu\nu\lambda} B_\nu \bar{p}_\lambda] \left(\frac{\lambda}{|\bar{p}|^3} \bar{\nabla}_\mu \mathcal{J}_n^{(0)''} - \frac{2}{|\bar{p}|^4} \bar{\nabla}_\mu \mathcal{J}_n^{(0)'} \right) + \frac{\lambda}{8|\bar{p}|^3} (\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}) \bar{\nabla}^\mu \bar{\nabla}^\nu \mathcal{J}_n^{(0)'}, \end{aligned}$$

Nonlocal effects in second-order chiral kinetic theory

Invariant phase space factor:

$$\sqrt{\omega}^{(2)} = -\frac{1}{2|\bar{p}|^5} \bar{\epsilon}^{\rho\mu\nu} \bar{p}_\rho E_\mu B_\nu - \frac{\lambda}{4} \bar{\nabla}^\mu \bar{\nabla}^\nu \frac{\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}}{|\bar{p}|^4}$$

Effective energy:

$$\begin{aligned} \epsilon^{(2)} = & \frac{1}{8|\bar{p}|^5} \left[(B \cdot \bar{p})^2 + B^2 \bar{p}^2 + (E \cdot \bar{p})^2 - \bar{p}^2 E^2 \right] + \frac{\lambda}{4|\bar{p}|^4} \bar{\epsilon}^{\mu\rho\sigma} E_\mu B_\rho \bar{p}_\sigma \\ & - \frac{\lambda}{8} \bar{\nabla}_\mu \frac{(E \cdot p) \bar{p}^\mu - \bar{p}^2 E^\mu - \lambda |\bar{p}| \bar{\epsilon}^{\mu\rho\sigma} B_\rho \bar{p}_\sigma}{|\bar{p}|^4} + \frac{1}{8} \bar{\nabla}^\mu \bar{\nabla}^\nu \frac{\bar{p}_\mu \bar{p}_\nu - \bar{p}^2 \Delta_{\mu\nu}}{|\bar{p}|^3} \end{aligned}$$

Effective four-velocity:

$$\begin{aligned} \dot{x}^{(2)\mu} \equiv & \frac{3|\bar{p}|(E \cdot p) E^\mu - 5|\bar{p}|(B \cdot p) B^\mu + \lambda(B \cdot p) \bar{\epsilon}^{\mu\alpha\beta} E_\alpha \bar{p}_\beta}{8|\bar{p}|^6} - \frac{11(B \cdot \bar{p})^2 + B^2 \bar{p}^2 + 5(E \cdot \bar{p})^2 - 11\bar{p}^2 E^2 + 4\lambda|\bar{p}| \bar{\epsilon}^{\nu\alpha\beta} E_\nu B_\alpha \bar{p}_\beta \bar{p}^\mu}{16|\bar{p}|^6} \\ & - \bar{\nabla}_\alpha \bar{\nabla}_\beta \frac{\bar{p}^\alpha \bar{p}^\beta \bar{p}^\mu - \bar{p}^2 \bar{p}^\mu \Delta^{\alpha\beta} + 2\bar{p}^2 \bar{p}^\mu \Delta^{\alpha\beta} - 2\bar{p}^2 \bar{p}^\alpha \Delta^{\mu\beta}}{16|\bar{p}|^5} + \lambda E_\alpha \bar{\nabla}_\beta \frac{2\bar{p}^\alpha \bar{p}^\beta \bar{p}^\mu + \bar{p}^2 \bar{p}^\mu \Delta^{\alpha\beta} - \bar{p}^2 \bar{p}^\alpha \Delta^{\mu\beta} - 2\bar{p}^2 \bar{p}^\beta \Delta^{\mu\alpha}}{8|\bar{p}|^6} + B_\alpha \bar{\nabla}_\beta \frac{\bar{\epsilon}^{\beta\alpha\nu} \bar{p}_\nu \bar{p}^\mu + 2\bar{\epsilon}^{\mu\beta\nu} \bar{p}_\nu \bar{p}^\alpha}{8|\bar{p}|^5} \end{aligned}$$

Second-order chiral kinetic equation in 7D

$$0 = \nabla_\mu \left[\sqrt{\omega}^{(0)} \dot{x}^{(0)\mu} \mathcal{J}_n^{(2)} + \sqrt{\omega}^{(1)} \dot{x}^{(0)\mu} \mathcal{J}_n^{(1)} + \sqrt{\omega}^{(0)} \dot{x}^{(1)\mu} \mathcal{J}_n^{(1)} + \sqrt{\omega}^{(1)} \dot{x}^{(1)\mu} \mathcal{J}_n^{(0)} + \sqrt{\omega}^{(0)} \dot{x}^{(2)\mu} \mathcal{J}_n^{(0)} + \sqrt{\omega}^{(2)} \dot{x}^{(0)\mu} \mathcal{J}_n^{(0)} \right]$$

$$\begin{aligned} 0 = & \sqrt{\omega}^{(0)} \dot{x}^{(0)\mu} \nabla_\mu \mathcal{J}_n^{(2)} + \sqrt{\omega}^{(1)} \dot{x}^{(0)\mu} \nabla_\mu \mathcal{J}_n^{(1)} + \sqrt{\omega}^{(0)} \dot{x}^{(1)\mu} \nabla_\mu \mathcal{J}_n^{(1)} + \sqrt{\omega}^{(1)} \dot{x}^{(1)\mu} \nabla_\mu \mathcal{J}_n^{(0)} + \sqrt{\omega}^{(0)} \dot{x}^{(2)\mu} \nabla_\mu \mathcal{J}_n^{(0)} + \sqrt{\omega}^{(2)} \dot{x}^{(0)\mu} \nabla_\mu \mathcal{J}_n^{(0)} \\ & + \left[\nabla_\mu, \sqrt{\omega}^{(0)} \dot{x}^{(0)\mu} \right] \mathcal{J}_n^{(2)} + \left[\nabla_\mu, \sqrt{\omega}^{(1)} \dot{x}^{(0)\mu} \right] \mathcal{J}_n^{(1)} + \left[\nabla_\mu, \sqrt{\omega}^{(0)} \dot{x}^{(1)\mu} \right] \mathcal{J}_n^{(1)} + \left[\nabla_\mu, \sqrt{\omega}^{(1)} \dot{x}^{(1)\mu} \right] \mathcal{J}_n^{(0)} + \left[\nabla_\mu, \sqrt{\omega}^{(0)} \dot{x}^{(2)\mu} \right] \mathcal{J}_n^{(0)} + \left[\nabla_\mu, \sqrt{\omega}^{(2)} \dot{x}^{(0)\mu} \right] \mathcal{J}_n^{(0)} \end{aligned}$$

$$\begin{aligned} 0 = & \dot{x}^{(0)\mu} \nabla_\mu \mathcal{J}_n^{(2)} - \frac{sB \cdot \bar{p}}{2|\bar{p}|^3} \hat{\bar{p}}^\mu \bar{\nabla}_\mu \mathcal{J}_n^{(1)} - \frac{\lambda s}{2|\bar{p}|^3} \bar{\epsilon}^{\mu\rho\sigma} E_\rho \bar{p}_\sigma \bar{\nabla}_\mu \mathcal{J}_n^{(1)} \\ & + \frac{3(B \cdot \bar{p})^2 - \bar{p}^2 B^2 - 5(E \cdot \bar{p})^2 - \bar{p}^2 E^2 - 4\lambda |\bar{p}| \bar{\epsilon}^{\nu\rho\sigma} \bar{p}_\nu E_\rho B_\sigma}{8|\bar{p}|^6} \lambda \hat{\bar{p}}^\mu \bar{\nabla}_\mu \mathcal{J}_n^{(0)} + \frac{(B \cdot \bar{p})}{4|\bar{p}|^6} \bar{\epsilon}^{\mu\rho\sigma} E_\rho \bar{p}_\sigma \bar{\nabla}_\mu \mathcal{J}_n^{(0)} - \frac{3(E \cdot \bar{p})}{4|\bar{p}|^5} \lambda E^\mu \bar{\nabla}_\mu \mathcal{J}_n^{(0)} \\ & + \frac{\bar{\epsilon}^{\nu\rho\sigma} \bar{p}_\rho B_\sigma}{4|\bar{p}|^4} \lambda \hat{\bar{p}}^\mu \bar{\nabla}_\mu \bar{\nabla}_\nu \mathcal{J}_n^{(0)} + \frac{2(E \cdot \bar{p})(\bar{p}^2 \Delta^{\mu\nu} - \bar{p}^\mu \bar{p}^\nu) + \bar{p}^2 [\bar{p}^\mu E^\nu - (E \cdot \bar{p}) \Delta^{\mu\nu}]}{4|\bar{p}|^6} \bar{\nabla}_\mu \bar{\nabla}_\nu \mathcal{J}_n^{(0)} - \frac{\bar{p}_\rho \bar{p}_\sigma - \bar{p}^2 \Delta_{\rho\sigma}}{8|\bar{p}|^4} \lambda \hat{\bar{p}}^\mu \bar{\nabla}_\mu \bar{\nabla}^\rho \bar{\nabla}^\sigma \mathcal{J}_n^{(0)} \end{aligned}$$

Other relevant works from Wigner function (top-down)



PUBLISHED FOR SISSA BY SPRINGER

RECEIVED: December 24, 2020

REVISED: February 27, 2021

ACCEPTED: April 7, 2021

PUBLISHED: May 4, 2021

Second order chiral kinetic theory under gravity and antiparallel charge-energy flow

Tomoya Hayata,^a Yoshimasa Hidaka^{b,c,d} and Kazuya Mameda^e

^aDepartments of Physics, Keio University,
4-1-1 Hiyoshi, Kanagawa 223-8521, Japan

^bKEK Theory Center,
Tsukuba 305-0801, Japan

^cGraduate University for Advanced Studies (Sokendai),
Tsukuba 305-0801, Japan

$$\mathcal{R}^\mu(x, p) = \frac{1}{2} \text{tr} \left[\gamma^\mu \frac{1 + \gamma^5}{2} W(x, p) \right],$$

$$W_{ab}(x, p) = \int d^4y \sqrt{-g(x)} e^{-ip \cdot y / \hbar} \langle \bar{\psi}_b(x, y/2) \psi_a(x, -y/2) \rangle,$$

$$\begin{aligned} \mathcal{R}_\mu = & 2\pi \delta(p^2) \left[p_\mu \left(f_{(0)} + \hbar f_{(1)} + \hbar^2 f_{(2)} \right) + \hbar \Sigma_{\mu\nu}^n D^\nu f_{(0)} + \hbar^2 \Sigma_{\mu\nu}^u D^\nu f_{(1)} \right] \\ & + 2\pi \hbar^2 \frac{1}{p^2} \left[-p_\mu Q \cdot p + 2p^\nu \left(T_{[\mu} p_{\nu]} + S_{\alpha\mu\nu} p^\alpha \right) \right] \delta(p^2) f_{(0)} \\ & + 2\pi \hbar^2 \frac{\delta(p^2)}{p^2} \left[\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} p^\nu D^\rho \Sigma_n^{\sigma\lambda} D_\lambda - \Sigma_{\mu\nu}^u \left(\frac{1}{2} \tilde{R}^{\alpha\beta\nu\rho} p_\rho p_\alpha \partial_\beta^p + p \cdot D \Sigma_n^{\nu\rho} D_\rho \right) \right] f_{(0)} \end{aligned}$$

PHYSICAL REVIEW D **108**, 016001 (2023)

Nonlinear chiral kinetic theory

Kazuya Mameda[✉]

Department of Physics, Tokyo University of Science, Tokyo 162-8601, Japan
and RIKEN iTHEMS, RIKEN, Wako 351-0198, Japan



(Received 9 May 2023; accepted 14 June 2023; published 5 July 2023)

From quantum field theory, we derive the chiral kinetic theory involving nonlinear quantum corrections coupled with spacetime-dependent electromagnetic fields and fluid velocity gradients. An equilibrium Wigner function determined by the kinetic equation verifies the nondissipativeness of the charge induced by the magneto-vortical coupling. We reveal that this nonlinear chiral kinetic theory is consistent with the one-loop Euler–Heisenberg effective theory, indicating an indirect evidence of the trace anomaly in the kinetic theory. We also argue a potential issue on the regularization, and demonstrate the availability of the point-splitting regularization in the nonlinear chiral kinetic theory.

DOI: 10.1103/PhysRevD.108.016001

$$W = \mathcal{F} + i\gamma^5 \mathcal{P} + \gamma^\mu \mathcal{V}_\mu + \gamma^5 \gamma^\mu \mathcal{A}_\mu + \frac{1}{2} \sigma^{\mu\nu} S_{\mu\nu},$$

$$\mathcal{R}(x, p) := \frac{1}{2} \text{tr} [\gamma^\mu P_R W(x, p)] \quad P_R := \frac{1}{2} (1 + \gamma^5)$$

$$\begin{aligned} \mathcal{R}_\mu^{(2)} = & 2\pi \delta(p^2) \left[p_\mu f_{(2)} + \left(\Sigma_{\mu\nu}^u \Delta^\nu - \frac{1}{p^2} \tilde{F}_{\mu\nu} p^\nu \right) f_{(1)} - \Sigma_{\mu\nu}^u \varepsilon^{\nu\rho\sigma\lambda} \Delta_\rho \frac{n_\sigma}{2p \cdot n} \Delta_\lambda f_{(0)} \right] \\ & + 2\pi \frac{\delta(p^2)}{p^2} \left(\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} p^\nu \Delta^\rho + \frac{p_\mu p^\nu}{p^2} \tilde{F}_{\nu\sigma} - \tilde{F}_{\mu\sigma} \right) \left(\Sigma_n^{\sigma\lambda} \Delta_\lambda - \frac{1}{p^2} \tilde{F}^{\sigma\lambda} p_\lambda \right) f_{(0)} \\ & + 2\pi \frac{\delta(p^2)}{p^2} \Sigma_{\mu\nu}^u \left[\Delta_\alpha \Sigma_n^{\alpha\nu} + \frac{n_\alpha}{p \cdot n} \tilde{F}^{\alpha\nu} + \frac{1}{p^2} \tilde{F}^{\nu\lambda} p_\lambda \right] p \cdot \Delta f_{(0)}, \\ & + \frac{2\pi}{p^2} [-p_\mu Q \cdot p + 2p^\nu Q_{[\mu} p_{\nu]}] f_{(0)} \delta(p^2) \end{aligned}$$

Other relevant works from semiclassical approach (bottom-up)

PHYSICAL REVIEW B 95, 205141 (2017)

Second-order chiral kinetic theory: Chiral magnetic and pseudomagnetic waves

E. V. Gorbar,^{1,2} V. A. Miransky,^{3,4} I. A. Shovkovy,^{5,6} and P. O. Sukhachov³

¹Department of Physics, Taras Shevchenko National Kiev University, Kiev 03680, Ukraine

²Bogolyubov Institute for Theoretical Physics, Kiev 03680, Ukraine

³Department of Applied Mathematics, Western University, London, Ontario, Canada N6A 5B7

⁴Department of Physics and Astronomy, Western University, London, Ontario, Canada N6A 3K7

⁵College of Integrative Sciences and Arts, Arizona State University, Mesa, Arizona 85212, USA

⁶Department of Physics, Arizona State University, Tempe, Arizona 85287, USA

(Received 17 February 2017; revised manuscript received 7 April 2017; published 26 May 2017)

The consistent chiral kinetic theory accurate to the second order in electromagnetic and pseudoelectromagnetic fields is derived for a relativistic matter with two Weyl fermions. By making use of such a framework, the properties of longitudinal collective excitations, which include both chiral magnetic and chiral pseudomagnetic waves, are studied. It is shown that the proper treatment of dynamical electromagnetism transforms these gapless waves into chiral (pseudo)magnetic plasmons, whose Langmuir (plasma) gap receives corrections quadratic in both magnetic and pseudomagnetic fields.

DOI: 10.1103/PhysRevB.95.205141

$$H = \lambda v_F (\mathbf{p} \cdot \boldsymbol{\sigma})$$

$$\Omega_\lambda \equiv \Omega_\lambda^{(0)} + \Omega_\lambda^{(1)} = \lambda \eta \hbar \frac{\mathbf{p}}{2p^2} + \frac{e\hbar^2}{4p^4} \left\{ \frac{2}{c} \hat{\mathbf{p}}(\hat{\mathbf{p}} \cdot \mathbf{B}_\lambda) - \frac{1}{c} \mathbf{B}_\lambda + \frac{2\eta}{v_F} [\mathbf{E}_\lambda \times \hat{\mathbf{p}}] \right\}.$$

$$\epsilon_p = \eta v_F p - \lambda \frac{e\hbar v_F}{2cp} (\mathbf{B}_\lambda \cdot \hat{\mathbf{p}}) + \frac{e^2 \hbar^2}{4cp^3} \left\{ \frac{\eta v_F}{4c} [2B_\lambda^2 - (\mathbf{B}_\lambda \cdot \hat{\mathbf{p}})^2] - (\mathbf{B}_\lambda \cdot [\mathbf{E}_\lambda \times \hat{\mathbf{p}}]) \right\}.$$

$$\frac{\partial f_\lambda}{\partial t} + \frac{1}{1 + \frac{e}{c} (\mathbf{B}_\lambda \cdot \boldsymbol{\Omega}_\lambda)} \left[\left(e \tilde{\mathbf{E}}_\lambda + \frac{e}{c} (\mathbf{v} \times \mathbf{B}_\lambda) + \frac{e^2}{c} (\tilde{\mathbf{E}}_\lambda \cdot \mathbf{B}_\lambda) \boldsymbol{\Omega}_\lambda \right) \cdot \frac{\partial f_\lambda}{\partial \mathbf{p}} + \left(\mathbf{v} + e (\tilde{\mathbf{E}}_\lambda \times \boldsymbol{\Omega}_\lambda) + \frac{e}{c} (\mathbf{v} \cdot \boldsymbol{\Omega}_\lambda) \mathbf{B}_\lambda \right) \cdot \frac{\partial f_\lambda}{\partial \mathbf{r}} \right] = 0,$$

e-Print:2509.15731

RIKEN-iTHEMS-Report-25

Quantum Metric Corrections to Liouville's Theorem and Chiral Kinetic Theory

Kazuha Mameda^{1,2} and Naoki Yamamoto³

¹Department of Physics, Tokyo University of Science, Tokyo 162-8601, Japan

²RIKEN iTHEMS, RIKEN, Wako, Saitama 351-0198, Japan

³Department of Physics, Keio University, Yokohama 223-8522, Japan

Quasiparticles may possess not only Berry curvature but also a quantum metric in momentum space. We develop a canonical formalism for such quasiparticles based on the Dirac brackets, and demonstrate that quantum metric modifies the phase-space density of states at $\mathcal{O}(\hbar^2)$, leading to corrections to Liouville's theorem, kinetic theory, and related physical quantities. In particular, we show that, in the presence of an inhomogeneous electric field, quantum metric induces corrections to the energy density and energy current. Applied to chiral fermions, this framework provides a nonlinear extension of chiral kinetic theory consistent with quantum field theory. Our work paves the way to potential applications of the quantum metric in high-energy physics and astrophysics.

$$L = \mathbf{p} \cdot \dot{\mathbf{x}} - \epsilon - \hbar \mathbf{a} \cdot \dot{\mathbf{p}} - \frac{\hbar^2}{2} \dot{p}_i \dot{p}_j G_{ij},$$

$$b_i = \frac{\hat{p}_i}{2|\mathbf{p}|^2}, \quad G_{ij} = \frac{1}{4|\mathbf{p}|^3} (\delta_{ij} - \hat{p}_i \hat{p}_j),$$

$$\sqrt{\omega} = 1 + \hbar^2 \partial_i E_j G_{ij}, \quad \epsilon'_p = \epsilon_p - \frac{\hbar^2}{2} E_i E_j G_{ij}.$$

$$\left[\partial_t + (\mathbf{v}' + \hbar \mathbf{E} \times \mathbf{b}' + \hbar^2 \mathbf{V}) \cdot \boldsymbol{\nabla} + \mathbf{E} \cdot \boldsymbol{\nabla}_p \right] n_p = 0,$$

$$\mathbf{v}' = \boldsymbol{\nabla}_p \epsilon'_p \quad V_k = -G_{kj} v_i \partial_i E_j.$$

Full second-order chiral kinetic theory from AI

Second-order chiral kinetic theory in inhomogeneous electromagnetic fields in a lower-order-equation normal form

Anonymous Author(s)¹

¹*Affiliation to be supplied by the authors*

We derive the collisionless chiral kinetic equation through second order for a massless fermion in a general prescribed Abelian electromagnetic field and give an explicit lower-order-equation normal form. The Wigner operators retain $\Pi^{(2)} \sim \partial F \partial_p^2$, $G^{(2)} \sim \partial^2 F \partial_p^3$, and the noncommutativity of phase-space derivatives. After the finite branch residues, every time derivative acting on f_0 or f_1 is recursively eliminated with the zeroth- and first-order kinetic hierarchy, whereas the leading $\partial_t \partial_x$ and $\partial_t E$ terms remain. Only Bianchi identities are used; no integration by parts or Maxwell source equation enters. For $r > 0$ the explicit second-order f_0 operator has 56 undifferentiated-field and 46 field-jet vector-tensor monomials. Exact Cartesian re-expansion gives 1322 canonical coefficients, and a differential-ideal certificate proves the EOM reduction coefficientwise. Compact electric and magnetic profiles give six explicit benchmarks and demonstrate that EOM-first and stationarity-first normal forms need not commute off shell. Direct coefficient regression to the published uniform Eq. (3.54) is established for $\lambda = +1$; a unified two-branch comparison is left open pending the Paper's triangular distribution redefinitions.

$$0 = \mathcal{K}_0[f_2] + \widetilde{\mathcal{K}}_1[f_1] + \widetilde{\mathcal{K}}_{2,\text{h}}[f_0] + \Delta \widetilde{\mathcal{K}}_{2,\text{inh}}[f_0].$$

$$\mathcal{K}_0[f] = \frac{1}{2} \left[\lambda \partial_t + \frac{\mathbf{p}}{r} \cdot \nabla_{\mathbf{x}} + \left(\lambda \mathbf{E} + \frac{\mathbf{p} \times \mathbf{B}}{r} \right) \cdot \nabla_{\mathbf{p}} \right] f,$$

$$\begin{aligned}\widetilde{\mathcal{K}}_1[f] = & \frac{s}{4} \left[\frac{\lambda(\mathbf{p} \cdot \mathbf{B})\mathbf{p}}{r^4} \cdot \nabla_x f - \frac{\mathbf{p} \times \mathbf{E}}{r^3} \cdot \nabla_x f \right. \\ & + \frac{(\mathbf{E} \cdot \mathbf{B})\mathbf{p}}{r^3} \cdot \nabla_p f - \frac{(\mathbf{p} \cdot \mathbf{B})\mathbf{E}}{r^3} \cdot \nabla_p f \\ & \left. + \frac{\lambda(\mathbf{p} \cdot \mathbf{B})(\mathbf{p} \times \mathbf{B})}{r^4} \cdot \nabla_p f + \frac{\lambda(\nabla_x \mathbf{B})\mathbf{p}}{r^2} \cdot \nabla_p f \right]\end{aligned}$$

$$\widetilde{\mathcal{K}}_{2,h} = \sum_{a=1}^{56} \widetilde{\mathcal{M}}_a, \quad \Delta \widetilde{\mathcal{K}}_{2,\text{inh}} = \sum_{a=1}^{46} \Delta \widetilde{\mathcal{M}}_a.$$

Appendix C: Explicit field-jet EOM-reduced operator

The next 46 monomials contain first or second field jets, including the retained $\partial_i E$ structures. They vanish when $\partial_\alpha F_{\mu\nu} = 0$. Faraday's law has removed $\partial_i B$, but no Ampère or Gauss equation has been imposed. As in the preceding list, there is no implicit product derivative or integration by parts. The exact abstract-index support separates as follows; a first field jet may be spatial or temporal.

Coefficient class	Rows	Rows containing $\partial_i E$
∂F	13	4
$\partial^2 F$	5	0
$F \partial F$	28	0

$$\begin{aligned} \Delta \tilde{\mathcal{H}}_{2,\text{int}}[f_0] = & -\frac{1}{24} r^{-3} p_i \partial_i B_j B_k \partial_i^p \partial_j^p f_i \\ & + \frac{1}{8} r^{-5} p_i p_k p_i B_j \partial_i B_k \partial_i^p \partial_j^p f_0 \\ & + \frac{1}{24} r^{-3} p_i p_i B_k \partial_i B_j \partial_i^p \partial_j^p f_0 \end{aligned}$$

[illegible]

$$\begin{aligned}
& -\frac{1}{8} \epsilon^{-2} p_1 \epsilon_{ij} \partial_j^2 f_0 \\
& -\frac{1}{24} \epsilon^{-3} p_1 \epsilon_{ijk} \partial_j B_k \partial_i^2 f_0 \\
& -\frac{1}{12} \epsilon^{-3} p_1 \epsilon_{ijk} \partial_j B_k \partial_i^2 f_0 \\
& -\frac{1}{2} \epsilon^{-3} p_1 \epsilon_{ijk} \partial_j B_k \partial_i^2 f_0 \\
& +\frac{1}{8} \epsilon^{-4} p_1 \epsilon_{ijk} \epsilon_{lmn} \partial_l \partial_m \partial_n f_0 \\
& -\frac{1}{2} \epsilon^{-5} p_1 \epsilon_{ijk} \epsilon_{lmn} \partial_l \partial_m \partial_n B_k \partial_i^2 f_0
\end{aligned}$$

Component expansion of this list gives 689 keys. Together with the 568 keys from App. B, it reproduces the complete 1257-key EOM-reduced f_0 sector without a residual coefficient.

Full second-order chiral kinetic theory from AI

已经找到明显更紧凑的形式：原来的 56 项无场导数部分和 46 项场梯度部分，都可分别写成按一、二、三阶导数分组的三个块。六个块与原文逐项展开后的差，在 Mathematica 中均精确为零；检验保留了符号化的 λ ，没有假设均匀场、静态场或真空 Maxwell 方程。接下来整理可直接使用的 LaTeX 和复核脚本。

$$0 = \mathcal{K}_0[f_2] + \widetilde{\mathcal{K}}_1[f_1] + \widetilde{\mathcal{K}}_{2,h}[f_0] + \Delta \widetilde{\mathcal{K}}_{2,\text{inh}}[f_0].$$

$$\mathcal{K}_0[f] = \frac{1}{2} \left[\lambda \partial_t + \frac{\mathbf{p}}{r} \cdot \nabla_x + \left(\lambda \mathbf{E} + \frac{\mathbf{p} \times \mathbf{B}}{r} \right) \cdot \nabla_p \right] f,$$

$$\widetilde{\mathcal{K}}_1[f] = \frac{s}{4r^2} \mathfrak{N}[(\lambda b \mathbf{u} - \mathbf{u} \times \mathbf{E}) \cdot \mathbf{d} + \lambda r (\mathbf{G}^\top \mathbf{u}) \cdot \boldsymbol{\eta}] f.$$

$$\widetilde{\mathcal{K}}_{2,h}[f] = \mathfrak{N} \left[\sum_{m=1}^3 H_m \right] f, \quad \Delta \widetilde{\mathcal{K}}_{2,\text{inh}}[f] = \mathfrak{N} \left[\sum_{m=1}^3 J_m \right] f.$$

$$\mathfrak{N} \left[\sum_{\alpha,\beta} c_{\alpha\beta}(t, \mathbf{x}, \mathbf{p}) \xi^\alpha \eta^\beta \right] f := \sum_{\alpha,\beta} c_{\alpha\beta}(t, \mathbf{x}, \mathbf{p}) \partial_x^\alpha \partial_p^\beta f.$$

$$H_1 = \frac{\chi L + 6e \mathbf{E} \cdot \mathbf{d} - 2\lambda b (\mathbf{u} \times \mathbf{E}) \cdot \mathbf{d}}{16r^4},$$

$$H_2 = -\frac{\lambda [e d^2 + L \mathbf{E} \cdot \mathbf{d} - 2e L^2] + L \mathbf{c} \cdot \mathbf{d}}{8r^3},$$

$$H_3 = \frac{L \mathbf{d} \cdot \mathbf{P}_\perp \mathbf{d}}{16r^2}.$$

$$J_1 = \mathbf{A} \cdot \mathbf{d} + \mathbf{R} \cdot \boldsymbol{\eta},$$

$$J_2 = \frac{1}{24r^2} \left\{ L(\boldsymbol{\omega} - 3\mathbf{u} \times \mathbf{h}) \cdot \boldsymbol{\eta} - 2\mathbf{d} \cdot [\mathbf{u} \times (\mathbf{G}\boldsymbol{\eta}) + \boldsymbol{\eta} \times \mathbf{h}] \right. \\ \left. - 3\lambda (\mathbf{P}_\perp \mathbf{d}) \cdot (\mathbf{H}\boldsymbol{\eta}) \right\},$$

$$J_3 = -\frac{(\mathbf{P}_\perp \mathbf{d}) \cdot [\boldsymbol{\eta} \times (\mathbf{G}\boldsymbol{\eta})]}{24r} - \frac{1}{48} \boldsymbol{\eta} \cdot (\boldsymbol{\eta} \cdot \nabla_x)^2 (\lambda \mathbf{E} + \mathbf{u} \times \mathbf{B}).$$

$$\mathbf{A} = \frac{\boldsymbol{\omega} - 3\mathbf{P}_\perp \mathbf{T} + 3\lambda \mathbf{u} \text{tr}(\mathbf{P}_\perp \mathbf{H})}{24r^3},$$

$$\mathbf{R} = \frac{1}{8r^3} \left\{ -\mathbf{P}_\perp \boldsymbol{\ell} - (\mathbf{u} \cdot \mathbf{h}) \mathbf{B} + 2b\mathbf{g} + \lambda [\mathbf{u} \times (\mathbf{G}\mathbf{E} + 2e\mathbf{h}) + 2\mathbf{h} \times \mathbf{E} - e\boldsymbol{\omega}] \right\} + \frac{2D_u \boldsymbol{\omega} - \mathbf{u} \times (\nabla_x^2 \mathbf{B} + D_u^2 \mathbf{B})}{16r^2}.$$

$$\mathbf{G}_{ij} = \partial_j B_i, \quad \mathbf{H}_{ij} = \partial_j E_i, \quad \mathbf{h} = \mathbf{G}\mathbf{u}, \quad \mathbf{g} = \mathbf{G}^\top \mathbf{u}, \quad \boldsymbol{\ell} = \mathbf{G}^\top \mathbf{B},$$

$$\boldsymbol{\omega} = \nabla_x \times \mathbf{B}, \quad \mathbf{T} = \partial_t \mathbf{E}, \quad D_u = \mathbf{u} \cdot \nabla_x.$$

Summary and Outlook

- A second-order chiral kinetic theory in 7D phase space under constant electromagnetic fields has been derived from Wigner function formalism.
- Non-local contributions regarding the invariant phase space factor, effective energy, four-velocity and kinetic equations are derived.

- Bottom-up and top-down methods should complement each other in the study of quantum kinetic theory
- While a second-order chiral kinetic theory under varying electromagnetic fields has been calculated via AI, the verification of these results is currently under way.

Thank you for your attention !